INFORMATION AND CONTROL 21, 48–55 (1972)

Some Comments on Neighborhood Size for Tessellation Automata

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The simplest possible neighborhood interconnection patterns that can be constructed from both partition and cover blocking preserving behavioral isomorphism are established for a broad class of two-dimensional arrays.

Introduction

The first person to introduce and use essentially the idea of a "cover blocking" on a array appears to be Cole (1966) in his work on real-time processing in arrays.

In their contributions to the theory of processing in arrays, Korsaraju (1968) and Hamacher (1971) made essential use of what we call "blocking" for neighborhood standardization preserving processing capabilities. These ideas were systematically studied in Smith (1971) and in Yamada–Amoroso (1971).

It was known that partition blockings on two-dimensional tessellation arrays can reduce any neighborhood structure to one with just seven components preserving behavioral isomorphism. We show here that this is the limit of what can be done with partition blockings. It was also known that cover blockings yield a further reduction, but the fact that the reduction is to just three components seems new. This is then established as being the limit for cover blockings.

We assume the reader is not seeing the concept of a tessellation automaton for the first time. The brief review below is intended mainly to establish the notation which follows that introduced in Yamada–Amoroso (1969, 1971). Sections I, V, VII, and VIII of the last reference are most relevant.

Preliminaries

A *tessellation automaton* (TA) consists of a finite nonempty set A (the *state alphabet*), the set E^d of all d-tuples of integers (the *tessellation array*), and X, an n-tuple of distinct d-tuples of integers (the *neighborhood structure*). Any mapping $c : E^d \to A$ is called an (array) *configuration*. C will denote the set of all such mappings. If $i \in E^d$, and $X = (\xi_1, ..., \xi_n)$, when $N(X, i) = (i + \xi_1, ..., i + \xi_n)$ is the *neighborhood* of "cell" i relative to X. For any $c \in C$, $c(N(X, i))$ is defined as

$$(c(i + \xi_1), ..., c(i + \xi_n)).$$

A mapping $\tau : C \to C$ defined from a given (local) map $\sigma : A^n \to A$, by:

$$\tau(c) = c' \Leftrightarrow \text{for any } i \in Z^d, \qquad c'(i) = \sigma(c(N(X, i)),$$

is called a *parallel* map. For each TA, a set I of such parallel maps is also specified.

Cells $i, j \in E^d$ are called *immediate neighborhood related* with respect to $X = (\xi_1, ..., \xi_n)$, if $j = i + \xi_k$ or $j = i - \xi_k$ for some component ξ_k of X. Cells i, j are called *neighborhood related* (an equivalence relation) if $i = j$ or there is a sequence of cells $k_0, k_1, ..., k_m$ ($m \geqslant 1$) such that $i = k_0$, $j = k_m$ and k_q and k_{q+1} are neighborhood related for each q, $0 \leqslant q < m$. The partition $\{\Lambda_0, \Lambda_1, ...\}$ on E^d defined by this relation is called a *lamination* of the tessellation array. A TA is called *nonlaminated* if the lamination consists of only one equivalence class.

Let C_1 and C_2 be the sets of (array) configurations for two nonlaminated TA M_1 and M_2 of the same *dimension* d, i.e., both have tessellation arrays E^d, and let their parallel transformation sets be I_1 and I_2. An ordered pair of mappings

$$\mu_b = (\mu_c, \mu_\tau)$$

is called a *behavioral homomorphism* from M_1 into M_2 if $\mu_c : C_1 \to C_2$ and $\mu_\tau : I_1 \to I_2$ are such that for any $c_1 \in C_1$ and any $\tau_1 \in I_1$,

$$\mu_c(\tau_1(c_1)) = \mu_\tau(\tau_1)(\mu_c(c_1)).$$

If μ_b is a behavioral homomorphism from M_1 into M_2, if each component is bijective, and if $\mu_b^{-1} = (\mu_c^{-1}, \mu_\tau^{-1})$ is a behavioral homomorphism from M_2 into M_1, then M_1 and M_2 are called behaviorally *isomorphic.*

Consider an arbitrary submodule Λ_0 of E^d, and let $\{\Lambda_0, \Lambda_1, ...\}$ be the partition determined by the quotient module E^d/Λ_0. A *kernel block* with respect to Λ_0, will be any subset of E^d that satisfies (a) and (b) below:

(a) $O^d \in K_0$,

(b) For each $\Lambda_k \in E^d/\Lambda_0$, the cardinality of $\Lambda_k \cap K_0 \geqslant 1$ and finite.

For any submodule Λ_0 of E^d and any kernel block K_0 with respect to Λ_0, define

$$B(\Lambda_0, K_0) = \{K_j \mid K_j = K_0 + j \text{ for some } j \in \Lambda_0\},$$

where

$$K_0 + j = \{i + j \mid i \in K_0\}.$$

$B(\Lambda_0, K_0)$ is called a *cover blocking* of E^d and the subset elements are called blocks. If for each $\Lambda_k \in E^d/\Lambda_0$, $\Lambda_k \cap K_0 = 1$, then $B(\Lambda_0, K_0)$ is a partition on E^d and is called a *partition blocking*.

All the concepts just reviewed were introduced and studied at some length in Yamada–Amoroso (1969, 1971).

Theorem 5 and 6 in Section VIII of Yamada–Amoroso (1971) established that for any two-dimensional TA M_1 a behaviorally isomorphic two- dimensional TA M_2 could be constructed from a blocked structure where the neighborhood of M_2 has five components, or seven components if a partition blocking were employed. These bounds were conjectured to be minimum. Indeed, we shall establish that for certain two-dimensional TA (in fact, for any TA with a Moore neighborhood structure and a total parallel transformation set) no behaviorally isomorphic TA with a neighborhood of fewer than seven components can be constructed from a blocked structure arising from a *partition* blocking with a "connected" kernel block. However, we shall establish (for two-dimensional arrays) that for any TA, a behaviorally isomorphic TA can be constructed from a blocked structure arising from a (nonpartition) *cover* blocking and has a neighborhood structure with only *three* components. This is then established as being minimum for arbitrary blockings.

Just how a behaviorally isomorphic TA can be constructed from a blocking on a given TA by means of a blocked structure is too tedious to repeat here. The details are in Section VII of Yamada–Amoroso (1971) for the arbitrary

d-dimensional case, but we shall limit our attention to two-dimensional arrays exclusively.

THE LIMITATION ON PARTITION BLOCKING

The sequence of definitions and lemmas stated below should be sufficient to trace the reasoning used to establish the main result of this section, Corollary 9. Further details can be found in Guilfoyle (1971).

For $x = (x_1, x_2) \in E^2$, the set of ordered pairs of integers, define *norm* of x, $\| x \|$, to be $\max\{| x_1 |, | x_2 |\}$. For K a finite nonempty subset of E^2 (a *block*) and for $x \in E^2$, define the *distance* from x to K, $d(x, K) = \min_{k \in K} \| x - k \|$. The *distance* between two blocks K_1, K_2 is defined by $d(K_1, K_2) = \min_{k_1 \in K_1} d(k_1, K_2)$. Two blocks will be called *neighbors* if the distance between them is one. A block will be said to be *connected* if it is not equal to the union of two blocks K_1, K_2 where $d(K_1, K_2) > 1$. Let K_0 denote any block containing the origin (a *kernel* block) and let $K_p = K_0 + p = \{k + p \mid k \in K_0\}$. The point p will be called the *center* of K_p.

Let $W_0 = \{w \in E^2 \mid d(K_w, K_0) = 1\}$, i.e., the set of all centers of blocks of form K_p neighboring K_0.

LEMMA 1. *For any K_0, W_0 is symmetric with respect to the origin ($p \in W_0$ implies $-p \in W_0$).*

For arbitrary subsets $A, B \subsetneq E^2$, A will be said to *enclose* B if every unbounded connected set intersecting B also intersects A.

LEMMA 2. *For arbitrary K_0 and W_0 as defined above, W_0 encloses K_0.*

LEMMA 3. $W_0 = \{w \mid d(w, \bigcup_{x \in K_0} K_{-x}) = 1\}$.

We must now show that for any connected K_0, there is a connected subset W_0' of W_0 enclosing K_0. For this purpose we now state a special case of a more general result (Corollary 2.61, p. 108 from Whyburn, 1942).

LEMMA 4. *If J_1 and J_2 are simple closed curves in the plane with I_1 and I_2 their respective sets of interior points, then $J_1 \cap J_2 \neq \varnothing$ implies that $J_1 \cup J_2$ contains a simple closed curve D such that $I_1 \cup I_2 \subsetneq$ interior D.*

We extend the distance concept to pairs of real numbers as follows. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in R^2, $d_R(x, y) = \max\{| x_1 - y_1 |, | x_2 - y_2 |\}$. For $x \in E^2$, let $J_x = \{y \in R^2 \mid d_R(y, x) = 1\}$.

PROPOSITION 5. *For any $x \in E^2$, J_x is a simple closed curve.*

If K_0 is an arbitrary connected kernel block, then $\bigcup_{x \in K_0} K_{-x} = \{x_1, ..., x_n\}$ is also connected and the points can be ordered so that the corresponding sequence of curves $J_{x_1}, ..., J_{x_n}$ are such that $J_{x_i} \cap J_{x_{i+1}} \neq \varnothing$ for all i, $1 \leqslant i < n$. Using Lemmas 3 and 4 inductively, we can show:

LEMMA 6. *For any connected (in E^2) K_0, W_0 contains a connected subset W_0' enclosing K_0.*

LEMMA 7. *For connected K_0, let W_0' be a connected subset of W_0 enclosing K_0, then for any $p \in W_0'$ there exists a $q \neq p$ or $-p$, in $W_0' \cap W_p'$, where $W_p' = W_0' + p$.*

Proof. It is easy to argue the existence of points in W_p', one enclosed by W_0' and one not enclosed by W_0'. From connectivity we could conclude that $W_0' \cap W_p' \neq \varnothing$. Since $0 \notin W_0$, $p \notin W_p$, hence $\notin W_p'$. If $\{-p\} = W_0' \cap W_p'$, then $-p = w + p$, $w \in W_0$. This implies $-2p \in W_0$ and by Lemma 1, $2p \in W_0$. Instead of starting with p and forming W_p, we could therefore have started with $2p$ and formed W_{2p} and argued the existence of $4p \in W_0$. Continuing, we would have the contradiction that W_0 is infinite.

THEOREM 8. *Let $B(\Lambda_0, K_0)$ be any partition blocking of E^2 with K_0 connected. Then $\#(\Lambda_0 \cap W_0) \geqslant 6$.*

Proof. Let $p \in W_0$ be such that $K_p \in B(\Lambda_0, K_0)$. Let $W_p = W_0 + p$, then using Lemmas 6 and 7 and the connectedness of K_0, we have $W_0 \cap W_p \neq \varnothing$. There is a point x such that $x \notin K_0$, $x \notin K_p$, and $d(x, K_0) = d(x, K_p) = 1$. x must be in some $K_q \in B(\Lambda_0, K_0)$ where $q \in W_0 \cap W_p$. We therefore have three mutually neighboring translates K_p, K_q, K_0. The translates K_0, K_{q-p}, K_{-p} would also be mutually neighboring, and it is easy to verify that the points p, q, $-p$, $-q$, $p - q$, $q - p$ are all distinct and in $\Lambda_0 \cap W_0$.

It was shown in Yamada–Amoroso (1971) that any two-dimensional TA is behaviorally isomorphic to one with a Moore neighborhood structure, i.e., one where the neighbors are all cells not more than distance one away. If a partition blocking is introduced on an array with a Moore neighborhood structure, the next state of K_0 of the blocked structure would require information from each block distance one from K_0. We therefore have

COROLLARY 9. *There exists a two-dimensional* TA M_1 *such that any (behaviorally isomorphic)* TA M_2 *arising from a blocked structure on* M_1 *using* *a partition blocking would have at least seven components in its neighborhood structure.*

The Limitation on Cover Blocking

Let $B(\Lambda_0, K_0)$ be the (cover) blocking of E^2 where $K_0 = \{(x_1, x_2) \mid 0 \leqslant x_i \leqslant 8, i = 1, 2, \text{ and } x_1 + x_2 \leqslant 8\}$, and Λ_0 is generated by $(3, -1)$ and $(-1, 3)$. If we let $p = (3, -1)$, $q = (-1, 3)$, $r = (-2, -2)$, and

$$F(K_0) = \{x \in E^2 \mid d(x, K_0) = 1\},$$

then it is easy to see that $K_0 \cup F(K_0) \subset K_p \cup K_q \cup K_r$. This implies that if the above blocking were placed on an array interconnected by a Moore neighborhood structure, the next configuration of K_0 could be determined from the present configurations of K_p, K_q, and K_r. Since any TA is behaviorally isomorphic to one with a Moore neighborhood index (Proposition VIII.1, Yamada–Amoroso, 1971), the above remarks establish

Theorem 1. *For any two-dimensional* TA M_1 *there exists a two-dimensional* TA M_2 *constructible from* M_1 *by a blocked structure (using a cover blocking) and* M_2 *has a neighborhood structure with not more than three components.*

We now proceed to establish the minimality of the previous result. Let $p = (p_1, p_2) \neq (0, 0)$ be in E^2. Define

$$f_p\colon E^2 \to E$$

by: $f_p(x_1, x_2) = -p_2 x_1 + p_1 x_2$.

Lemma 2. *For any* $x, y \in E^2$ *and* $\alpha \in E$,

$$f_p(x + y) = f_p(x) + f_p(y),$$

$$f_p(\alpha x) = \alpha f_p(x).$$

Lemma *Let* $p \in E^2$, *then* $f_p(x + p) = f_p(x - p) = f_p(x)$ *for any* $x \in E^2$.

This implies $f_p(K_0) = f_p(K_p) = f_p(K_{-p})$. That is, the image of any kernel block under f_p is unaffected by any translation in the p-direction.

With K any block, $p \neq (0, 0)$, and $f_p(K) = \{f_p(x) \mid x \in K\}$, define $\underline{m}_p = \min f_p(K)$ and $\overline{m}_p = \max f_p(K)$.

LEMMA 4. *For any block K and any $p = (p_1, p_2) \neq (0, 0)$ in E^2, there exists a $\underline{y} \in F(K) = \{u \mid d(u, K) = 1\}$ such that $f_p(\underline{y}) < \underline{m}_p$, and there exists a $\bar{y} \in F(K)$ such that $f_p(\bar{y}) > \overline{m}_p$.*

Proof. Let $\underline{m}_p = f_p(x_p)$ where $x_p = (x_{p_1}, x_{p_2})$ and consider

$$y_{p_1} = x_{p_1} + \operatorname{sgn} p_2 ,$$

$$y_{p_2} = x_{p_2} - \operatorname{sgn} p_1 ,$$

where

$$\operatorname{sgn} a = \begin{cases} +1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0, \end{cases}$$

then

$$\begin{aligned} f_p(y_{p_1}, y_{p_2}) &= f_p(x_{p_1}, x_{p_2}) + f_p(\operatorname{sgn} p_2 , -\operatorname{sgn} p_1) \\ &= \underline{m}_p - p_2 \operatorname{sgn} p_2 - p_1 \operatorname{sgn} p_1 \\ &= \underline{m}_p - | p_2 | - | p_1 | < \underline{m}_p , \end{aligned}$$

since $| p_1 | + | p_2 | > 0$. Further, $\underline{y} = (y_{p_1}, y_{p_2}) \in F(K)$ since $f_p(\underline{y}) < \underline{m}_p$ and $\| \underline{y} - x_p \| = \max\{| \operatorname{sgn} p_1 |, | \operatorname{sgn} p_2 |\} = 1$. The case of $\bar{y}$ is similar.

THEOREM 5. *For arbitrary (cover) blocking $B(\Lambda_0, K_0)$ and for arbitrary p, $q \in \Lambda_0 - \{(0, 0)\}$, $F(K_0) \cup K_0 \nsubseteq K_p \cup K_q$.*

Proof. Suppose $F(K_0) \cup K_0 \subseteq K_p \cup K_q$. Let $f_p(\underline{x}) = \min f_p(K_0) = \underline{m}_p$ and $f_p(\bar{x}) = \max f_p(K_0) = \overline{m}_p$. From Lemma 4 there are $\underline{y}$ and $\bar{y}$ in $F(K_0)$ such that $f_p(\underline{y}) < \underline{m}_p$ and $f_p(\bar{y}) > \overline{m}_p$. Since $f_p(K_0) = f_p(K_p)$, this implies that $\underline{y}$ and $\bar{y} \notin K_p$. Suppose $\underline{y}, \bar{y} \in K_q$, then $\underline{y} - q \in K_0$ and $\bar{y} - q \in K_0$. Also, $\underline{m}_p \leqslant f_p(\underline{y} - q) = f_p(\underline{y}) - f_p(q) = \underline{m}_p - | p_1 | - | p_2 | - f_p(q)$; hence

$$0 < | p_1 | + | p_2 | \leqslant -f_p(q).$$

In the case of $\bar{y}$ we have

$$\overline{m}_p \geqslant f_p(\bar{y} - q) = f_p(\bar{y}) - f_p(q) = \overline{m}_p + | p_1 | + | p_2 | - f_p(q),$$

hence $0 < | p_1 | + | p_2 | \leqslant f_p(q)$. This contradiction establishes our result.

Concluding Remarks

If one does not wish to distinguish between configurations that differ only by a "shift" of the symbols in the array, then one is led from the concept of behavioral isomorphism to the concept of *weak* behavioral isomorphism. This topic was discussed in Yamada–Amoroso (1971) where the minimum neighborhood interconnections preserving this property were established. Smith (1971) has some similar results on this topic.

We would be interested in knowing if the connectivity condition on kernel blocks is necessary for our results—we conjecture it is not. A generalization of these results to arrays of arbitrary dimension would be nice to see.

Received: August 2, 1971

References

Cole, S. (1966), Real-Time computation by N-dimensional iterative arrays of finite-state machines, IEEE Conf. Rec. 7th Annual Symp. Switching and Automata, pp. 53–77.

Korsaraju, S. R. (1968) Computations on iterative automata, Ph.D. dissertation, University of Pennsylvania 1969.

Hamacher, V. C. (1971), Machine complexity versus interconnection complexity in iterative arrays, *IEEE Proc. Computers* **C-20**, 321–323.

Smith, A. R., III (1971), Cellular automata complexity trade-offs, *Information and Control* **18**, 466–482.

Yamada, H., and Amoroso, S. (1969), Tessellation automata, *Information and Control* **14**, 299–317.

Yamada, H., and Amoroso, S. (1971), Structural and behavioral equivalences of tessellation automata, *Information and Control* **18**, 1–31.

Guilfoyle, R. (1971), Minimum neighborhood properties of blocked tessellation automata, Ph.D. dissertation, Stevens Institute of Technology, Hoboken, NJ.

Whyburn, G. T. (1942), Analytic topology, Amer. Math. Soc. Colloq. Publ. No. 28, American Mathematical Society, Providence, RI.